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Ground states of the Heisenberg evolution operator in discrete three-dimensional spacetime and quantum discrete BKP equations

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Abstract
In this paper we consider three-dimensional quantum q-oscillator field theory without spectral parameters. We construct an essentially large set of eigenstates of evolution with unity eigenvalue of discrete time-evolution operator. All these eigenstates belong to a subspace of a total Hilbert space where an action of the evolution operator can be identified with quantized discrete BKP equations (synonym Miwa equations). The key ingredients of our construction are specific eigenstates of a single three-dimensional R-matrix. These eigenstates are boundary states for hidden three-dimensional structures of $\mathscr{U}_q\big(B_n^{(1)}\big)$ and $\mathscr{U}_q\big(D_n^{(1)}\big)$.

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Introduction

A quantum q-oscillator system [1, 2] in three-dimensional spacetime is the result of canonical quantization of a Hamiltonian form [16] of discrete three-wave equations [5, 6, 12].

In the most general form [12, 19], the discrete three-wave equations involve some extra parameters (spectral parameters in the quantum world) and correspond to a generic AKP-type hierarchy of integrable systems. There are two special choices of spectral parameters corresponding to the discrete-differential geometry [3, 15] of discrete conjugate nets (syn. quadrilateral nets) [4, 6, 10]—either circular nets (syn. orthogonal nets) in Euclidean space or ortho-chronous hyperbolic nets in Minkowski space. There are a lot of equations associated with discrete nets; here we mean equations for angular data (rotation coefficients) [1, 10]. Algebraically, circular and hyperbolic nets are distinguished by a signature of determinant of a rotation matrix.

For the latter case of hyperbolic nets the equations of motion admit two constraints reducing the number of degrees of freedom of the Cauchy problem twice. One constraint corresponds to discrete BKP equations (syn. Miwa equations) [14]. Discrete BKP equations appear in discrete differential geometry in many ways [11], the constraint for the hyperbolic net just clearly shows the reduction. The other constraint is a real form on the equations of motion (curiously, we discuss in fact six-wave equations, they become three-wave upon this reality condition).

The discrete three-wave equations give a well-posed Cauchy problem in (2 + 1)-dimensional discrete spacetime. The quantized discrete three-wave equations are the Heisenberg equations of motion defined by a discrete time-evolution operator. The principal question of quantum theory is the spectral problem for the evolution operator (Schrödinger equation).

For arbitrary spectral parameters providing the unitarity of evolution operator the spectral problem is rather complicated. In this paper we study the evolution operator for trivial spectral parameters corresponding in classics to the hyperbolic net. We consider both the Fock space and modular representations of q-oscillators. The quantum analogue of the above-mentioned constraints provides a definition of a subspace of total Hilbert space, and in this subspace we construct a large set (presumably infinite set) of eigenstates with unity eigenvalue of the evolution operator. Since we use in particular a quantum analogue of a dBKP constraint, we refer the resulting quantum theory to the quantum discrete BKP equations.

This paper is organized as follows. In section 1, we formulate the Cauchy problem in classics and the quantum Heisenberg equation of motion and give a formal definition of the evolution operator. Section 1 is a brief outline of [1, 2]. In section 2 we discuss the reductions in classics and in the quantum case. The subspace of Hilbert space and eigenvectors of the evolution operator are constructed in section 3.

1. q-oscillator field theory

1.1. Local Yang–Baxter and auxiliary tetrahedron equations

The most convenient form of the auxiliary problem providing the Hamiltonian equations of motion is the local Yang–Baxter equation.

Let $L_{\alpha\beta}[\mathcal{A}]$ be a matrix acting in the tensor product of two two-dimensional vector spaces V_α and V_β,

$$L_{\alpha,\beta}[\mathcal{A}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \boldsymbol{k} & \boldsymbol{a}^+ & 0 \\ 0 & \boldsymbol{a}^- & -\boldsymbol{k} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{A} = (\boldsymbol{k}, \boldsymbol{a}^\pm). \tag{1}$$

In classics, the fields $\mathcal{A}$ are constrained by

$$\boldsymbol{k}^2 + \boldsymbol{a}^+\boldsymbol{a}^- = 1 \tag{2}$$

and have the bracket

$$\{\boldsymbol{a}^+, \boldsymbol{a}^-\} = \boldsymbol{k}^2. \tag{3}$$

Being quantized, $\mathcal{A}$ is the q-oscillator algebra

$$\boldsymbol{k}\boldsymbol{a}^\pm = q^{\pm 1}\boldsymbol{a}^\pm\boldsymbol{k}, \qquad \boldsymbol{a}^+\boldsymbol{a}^- = 1 - q^{-1}\boldsymbol{k}^2, \qquad \boldsymbol{a}^-\boldsymbol{a}^+ = 1 - q\boldsymbol{k}^2. \tag{4}$$

The local Yang–Baxter equation is

$$L_{\alpha,\beta}[\mathcal{A}_1]L_{\alpha,\gamma}[\mathcal{A}_2]L_{\beta,\gamma}[\mathcal{A}_3] = L_{\beta,\gamma}[\mathcal{A}_3']L_{\alpha,\gamma}[\mathcal{A}_2']L_{\alpha,\beta}[\mathcal{A}_1'], \tag{5}$$

which defines the following map $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \to \mathcal{A}'_1 \times \mathcal{A}'_2 \times \mathcal{A}'_3$:

$$\begin{aligned} &\left(\boldsymbol{k}_2\boldsymbol{a}_1^{\pm}\right)' = \boldsymbol{k}_3\boldsymbol{a}_1^{\pm} + \boldsymbol{k}_1\boldsymbol{a}_2^{\pm}\boldsymbol{a}_3^{\mp}, \qquad && \left(\boldsymbol{a}_2^{\pm}\right)' = \boldsymbol{a}_1^{\pm}\boldsymbol{a}_2^{\pm} - \boldsymbol{k}_1\boldsymbol{k}_3\boldsymbol{a}_2^{\pm}, \\ &\left(\boldsymbol{k}_2\boldsymbol{a}_3^{\pm}\right)' = \boldsymbol{k}_1\boldsymbol{a}_3^{\pm} + \boldsymbol{k}_3\boldsymbol{a}_1^{\mp}\boldsymbol{a}_2^{\pm}, \end{aligned} \tag{6}$$

where in addition

$$(\boldsymbol{k}_1\boldsymbol{k}_2)' = \boldsymbol{k}_1\boldsymbol{k}_2, \qquad (\boldsymbol{k}_2\boldsymbol{k}_3)' = \boldsymbol{k}_2\boldsymbol{k}_3. \tag{7}$$

These formulae work both in classics and in the quantum case. In classics, upon condition (2) for $\boldsymbol{k}'_j$, map (6) preserves the symplectic structure (3). In the quantum case map (6) is an automorphism of the tensor cube of the q-oscillator algebra (4) and therefore for any irreducible representation of (4) there exists an operator R_{123} such that

$$\mathcal{A}'_j = \mathsf{R}_{123}\mathcal{A}_j\mathsf{R}_{123}^{-1}, \qquad j = 1, 2, 3, \tag{8}$$

and the local Yang–Baxter equation becomes the auxiliary tetrahedron equation

$$L_{\alpha,\beta}[\mathcal{A}_1]L_{\alpha,\gamma}[\mathcal{A}_2]L_{\beta,\gamma}[\mathcal{A}_3]\mathsf{R}_{123} = \mathsf{R}_{123}L_{\beta,\gamma}[\mathcal{A}_3]L_{\alpha,\gamma}[\mathcal{A}_2]L_{\alpha,\beta}[\mathcal{A}_1]. \tag{9}$$

The key feature of map (6) is that it is the square root of unity,

$$\mathsf{R}_{123}^2\mathcal{A}_j\mathsf{R}_{123}^{-2} \equiv \mathcal{A}_j, \tag{10}$$

and thus we are able to choose the overall normalization of R_{123} such that

$$\mathsf{R}_{123}^2 = 1. \tag{11}$$

Matrices L, equation (1), satisfy a free-fermions condition. The local Yang–Baxter equation is the free-fermions form of Korepanov zero curvature representation [12, 13] for the matrices of auxiliary linear problem (linear problem for discrete three-wave equations)

$$X_{\alpha\beta}[\mathcal{A}] = \begin{pmatrix} \boldsymbol{k} & \boldsymbol{a}^{+} \\ \boldsymbol{a}^{-} & -\boldsymbol{k} \end{pmatrix}. \tag{12}$$

In the discrete differential geometry, this is a matrix of rotation coefficients, the case of a hyperbolic net in Minkowski space is fixed by the condition $\det X = -1$.

1.2. Lattice equations of motion

Consider now a three-dimensional cubic lattice with basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$,

$$\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3, \qquad n_1, n_2, n_3 \in \mathbb{Z}. \tag{13}$$

Map (6) is the local form of equations of motion:

$$\mathcal{A}_j = \mathcal{A}_{j,\boldsymbol{n}} \Rightarrow \mathcal{A}'_j = \mathcal{A}_{j,\boldsymbol{n}+\boldsymbol{e}_j}, \qquad j = 1, 2, 3. \tag{14}$$

With the spacetime argument $\boldsymbol{n}$, equation (5) is

$$L_{\alpha,\beta}[\mathcal{A}_{1,\boldsymbol{n}}]L_{\alpha,\gamma}[\mathcal{A}_{2,\boldsymbol{n}}]L_{\beta,\gamma}[\mathcal{A}_{3,\boldsymbol{n}}] = L_{\beta,\gamma}[\mathcal{A}_{3,\boldsymbol{n}+\boldsymbol{e}_3}]L_{\alpha,\gamma}[\mathcal{A}_{2,\boldsymbol{n}+\boldsymbol{e}_2}]L_{\alpha,\beta}[\mathcal{A}_{1,\boldsymbol{n}+\boldsymbol{e}_1}], \tag{15}$$

and equations (6) become

$$\begin{aligned} &\boldsymbol{k}_{2,\boldsymbol{n}+\boldsymbol{e}_2}\boldsymbol{a}_{1,\boldsymbol{n}+\boldsymbol{e}_1}^{\pm} = \boldsymbol{k}_{3,\boldsymbol{n}}\boldsymbol{a}_{1,\boldsymbol{n}}^{\pm} + \boldsymbol{k}_{1,\boldsymbol{n}}\boldsymbol{a}_{2,\boldsymbol{n}}^{\pm}\boldsymbol{a}_{3,\boldsymbol{n}}^{\mp}, \\ &\boldsymbol{a}_{2,\boldsymbol{n}+\boldsymbol{e}_2}^{\pm} = \boldsymbol{a}_{1,\boldsymbol{n}}^{\pm}\boldsymbol{a}_{2,\boldsymbol{n}}^{\pm} - \boldsymbol{k}_{1,\boldsymbol{n}}\boldsymbol{k}_{3,\boldsymbol{n}}\boldsymbol{a}_{2,\boldsymbol{n}}^{\pm}, \\ &\boldsymbol{k}_{2,\boldsymbol{n}+\boldsymbol{e}_2}\boldsymbol{a}_{3,\boldsymbol{n}+\boldsymbol{e}_3}^{\pm} = \boldsymbol{k}_{1,\boldsymbol{n}}\boldsymbol{a}_{3,\boldsymbol{n}}^{\pm} + \boldsymbol{k}_{3,\boldsymbol{n}}\boldsymbol{a}_{1,\boldsymbol{n}}^{\mp}\boldsymbol{a}_{2,\boldsymbol{n}}^{\pm}, \end{aligned} \tag{16}$$

relation (7) provides in addition

$$\boldsymbol{k}_{1,\boldsymbol{n}+\boldsymbol{e}_1}\boldsymbol{k}_{2,\boldsymbol{n}+\boldsymbol{e}_2} = \boldsymbol{k}_{1,\boldsymbol{n}}\boldsymbol{k}_{2,\boldsymbol{n}}, \qquad \boldsymbol{k}_{2,\boldsymbol{n}+\boldsymbol{e}_2}\boldsymbol{k}_{3,\boldsymbol{n}+\boldsymbol{e}_3} = \boldsymbol{k}_{2,\boldsymbol{n}}\boldsymbol{k}_{3,\boldsymbol{n}}. \tag{17}$$

Since locally the equations of motion are given by the symplectic map or by quantum automorphism, lattice equations (16) constitute classical Hamiltonian equations or quantum Heisenberg evolution. The unique way to define the discrete time is

$$\tau = \tau(\boldsymbol{n}) = n_1 + n_2 + n_3, \tag{18}$$

so that all fields in the left-hand side of (15) correspond to time τ and all fields in the right-hand side correspond to one step forward time $\tau + 1$. A choice of space-like vectors is irrelevant. For instance, we can choose[1]

$$\boldsymbol{e}_\tau = \boldsymbol{e}_2, \qquad \boldsymbol{e}_x = \boldsymbol{e}_1 - \boldsymbol{e}_2, \qquad \boldsymbol{e}_y = \boldsymbol{e}_3 - \boldsymbol{e}_2, \tag{19}$$

so that

$$\boldsymbol{n} = \underbrace{n_1\boldsymbol{e}_x + n_3\boldsymbol{e}_y}_{\boldsymbol{r}} + \tau\boldsymbol{e}_\tau, \tag{20}$$

where $\boldsymbol{r}$ stands for the space-like position vector. This gives

$$\mathcal{A}_{j,\boldsymbol{n}} \equiv \mathcal{A}_{j,\boldsymbol{r}}(\tau), \tag{21}$$

and

$$\mathcal{A}_{1,\boldsymbol{n}+\boldsymbol{e}_1} = \mathcal{A}_{1,\boldsymbol{r}+\boldsymbol{e}_x}(\tau+1), \qquad \mathcal{A}_{2,\boldsymbol{n}+\boldsymbol{e}_2} = \mathcal{A}_{2,\boldsymbol{r}}(\tau+1), \qquad \mathcal{A}_{3,\boldsymbol{n}+\boldsymbol{e}_3} = \mathcal{A}_{3,\boldsymbol{r}+\boldsymbol{e}_y}(\tau+1), \tag{22}$$

so that equations (16) are precisely the discrete time Hamiltonian flow.

Equation (16) is a well-posed Cauchy problem for a finite size of a constant time discrete surface with periodical boundary conditions:

$$\boldsymbol{r} = n_1\boldsymbol{e}_x + n_3\boldsymbol{e}_y, \qquad n_1, n_3 \in \mathbb{Z}_N. \tag{23}$$

In the quantum case, the Heisenberg equations of motion are defined by the evolution operator,

$$\Phi(\tau+1) = U\Phi(\tau)U^{-1}, \qquad U = \exp(\mathrm{i}H). \tag{24}$$

In the form of intertwining (15) the evolution operator is defined by

$$L_{\alpha,\beta}[\mathcal{A}_{1,\boldsymbol{r}}]L_{\alpha,\gamma}[\mathcal{A}_{2,\boldsymbol{r}}]L_{\beta,\gamma}[\mathcal{A}_{3,\boldsymbol{r}}] = U L_{\beta,\gamma}[\mathcal{A}_{3,\boldsymbol{r}+\boldsymbol{e}_y}]L_{\alpha,\gamma}[\mathcal{A}_{2,\boldsymbol{r}}]L_{\alpha,\beta}[\mathcal{A}_{1,\boldsymbol{r}+\boldsymbol{e}_x}]U^{-1}. \tag{25}$$

The matrix element (or kernel) of the evolution operator can be expressed in terms of the matrix elements of the R-matrix (8). Let $|\sigma'\rangle$ and $\langle\sigma|$ denote conjugated bases in the representation space of the q-oscillator, $\sum|\sigma\rangle\langle\sigma| = 1$ or $\int|\sigma\rangle\langle\sigma| = 1$. Then the R-matrix is defined by its matrix element or kernel

$$\langle\sigma_1, \sigma_2, \sigma_3|\mathsf{R}|\sigma_1', \sigma_2', \sigma_3'\rangle. \tag{26}$$

An explicit form of matrix elements for the Fock space representation and the kernel for modular representation can be found in the appendix. The matrix element or the kernel of the evolution operator is then evidently

$$\langle\boldsymbol{\sigma}|U|\boldsymbol{\sigma}'\rangle = \prod_{\boldsymbol{r}\in\mathbb{Z}_N^2} \langle\sigma_{1,\boldsymbol{r}}, \sigma_{2,\boldsymbol{r}}, \sigma_{3,\boldsymbol{r}}|\mathsf{R}|\sigma'_{1,\boldsymbol{r}+\boldsymbol{e}_x}, \sigma'_{2,\boldsymbol{r}}, \sigma'_{3,\boldsymbol{r}+\boldsymbol{e}_y}\rangle. \tag{27}$$

The evolution operator is unitary when R is unitary. The local structure of the evolution operator corresponds to relativistic casuality, and thus we have the relativistic quantum field theory.

[1] In general, any pair of (n_1, n_2, n_3) can be chosen as space-like coordinates. All choices are equivalent up to translation operators.

A complete set of integrals of motion is produced by an auxiliary layer-to-layer transfer matrix. It is defined as follows:

$$T(x,y) = \underset{V_\alpha\times V_\beta\times V_\gamma}{\mathrm{Trace}} \left(D_\alpha(x) D_\beta(xy) D_\gamma(y) \prod_{n_1}^{\curvearrowright} \prod_{n_3}^{\curvearrowleft} L_{\alpha_{n_3}\beta_{n_2}}[\mathcal{A}_{1,r}] L_{\alpha_{n_3}\gamma_{n_1}}[\mathcal{A}_{2,r}] L_{\beta_{n_2}\gamma_{n_1}}[\mathcal{A}_{3,r}] \right). \tag{28}$$

Here we consider the tensor product of two-dimensional spaces,

$$V_\alpha = \bigotimes_{n_3\in\mathbb{Z}_N} V_{\alpha_{n_3}}, \qquad V_\beta = \bigotimes_{n_2\in\mathbb{Z}_N} V_{\beta_{n_2}}, \qquad V_\gamma = \bigotimes_{n_1\in\mathbb{Z}_N} V_{\gamma_{n_1}}, \tag{29}$$

so that matrix $L_{\alpha_{n_3},\beta_{n_2}}$ corresponds to the components $V_{\alpha_{n_3}} \times V_{\beta_{n_2}}$, etc. The ordered product in (28) is taken over n_1,

$$\prod_{n_1}^{\curvearrowright} f_{n_1} \stackrel{def}{=} f_0 f_1 f_2 \cdots f_{N-1}, \tag{30}$$

and over n_3,

$$\prod_{n_3}^{\curvearrowleft} f_{n_3} \stackrel{def}{=} f_{N-1} f_{N-2} \cdots f_1 f_0. \tag{31}$$

The index n_2 of V_β spaces is related to n_1 and n_3,

$$n_2 = -n_1 - n_3 \quad (\tau = 0). \tag{32}$$

Boundary matrices D are defined by

$$D_\alpha(x) = \bigotimes_{n_3\in\mathbb{Z}_N} D_{\alpha_{n_3}}(x), \qquad D_\alpha(x) = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \in \mathrm{End}(V_\alpha), \quad \text{etc.} \tag{33}$$

In the definition of ordered products $\mathbb{Z}_N^2$ invariance is broken, however the final trace over all auxiliary spaces restores the $\mathbb{Z}_N^2$ invariance of the transfer matrix (28). Due to the ordering of products, the same transfer matrix can be identically rewritten as

$$T(x,y) = \underset{V_\alpha\times V_\beta\times V_\gamma}{\mathrm{Trace}} \times \left(D_\alpha(x) D_\beta(xy) D_\gamma(y) \prod_{n_1}^{\curvearrowright} \prod_{n_3}^{\curvearrowleft} L_{\beta_{n_2}\gamma_{n_1}}[\mathcal{A}_{3,r+e_y}] L_{\alpha_{n_3}\gamma_{n_1}}[\mathcal{A}_{2,r}] L_{\alpha_{n_3}\beta_{n_2}}[\mathcal{A}_{1,r+e_x}] \right), \tag{34}$$

where $n_2 = -n_1 - n_3 - 1$.

Comparing now the definition of evolution operator (25) and equivalence of (28) and (34), we deduce

$$UT(x,y) = T(x,y)U, \tag{35}$$

i.e., the layer-to-layer transfer matrix generates the invariants of evolution,

$$T(x,y) = \sum_{a,b} x^a y^b T_{a,b}, \qquad 0 \leqslant a,b \leqslant 2N, \qquad |a-b| \leqslant N. \tag{36}$$

From the theory of fermionic tetrahedron equations (see [2, 17]), we know that the layer-to-layer transfer matrices commute, i.e. the set of $T_{a,b}$ constitutes a family of $3N^2$ independent commutative operators (in classics, quantities in involution)—the integrals of evolution.

Moreover, in classics the following equation

$$J(x,y) \stackrel{def}{=} \sum_{a,b} (-)^{a+b+ab} x^a y^b T_{a,b} = 0 \tag{37}$$

defines the spectral curve with genus $g \leqslant 3N^2 - 3N + 1$ for the evolution map [12].

The constant-time section of the three-dimensional cubic lattice is known as the kagome lattice. Operator (28) is the layer-to-layer transfer matrix on the kagome lattice. The evolution can be seen as a simultaneous shift of all β-lines on the kagome lattice [12, 13].

2. Constraints

2.1. Classical field theory

There are two selected constraints for general $\mathcal{A} = (\boldsymbol{k}, \boldsymbol{a}^{\pm})$, $\boldsymbol{k}^2 \equiv 1 - \boldsymbol{a}^+\boldsymbol{a}^-$, breaking the Hamiltonian structure (3) but preserved by map (6) and therefore by the equations of motion (16). They are

$$B: \quad \boldsymbol{a}^+ = 1 - \boldsymbol{k}, \qquad \boldsymbol{a}^- = 1 + \boldsymbol{k} \tag{38}$$

and

$$D: \quad \boldsymbol{a}^+ = \boldsymbol{a}^-. \tag{39}$$

Constraint 'B' results the map

$$\boldsymbol{k}_1' = \frac{\boldsymbol{k}_1\boldsymbol{k}_2}{\boldsymbol{k}_1 + \boldsymbol{k}_3 - \boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}, \qquad \boldsymbol{k}_2' = \boldsymbol{k}_1 + \boldsymbol{k}_3 - \boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3, \qquad \boldsymbol{k}_3' = \frac{\boldsymbol{k}_2\boldsymbol{k}_3}{\boldsymbol{k}_1 + \boldsymbol{k}_3 - \boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}. \tag{40}$$

This is the well-known representation of a discrete BKP equation [14, 11] as the map satisfying the functional tetrahedron equation [8, 9]. The substitution

$$\boldsymbol{k}_{1,n} = u\frac{\tau_{n+e_2}\tau_{n+e_3}}{\tau_n\tau_{n+e_2+e_3}}, \qquad \boldsymbol{k}_{2,n} = v\frac{\tau_n\tau_{n+e_1+e_3}}{\tau_{n+e_1}\tau_{n+e_3}}, \qquad \boldsymbol{k}_{3,n} = w\frac{\tau_{n+e_1}\tau_{n+e_2}}{\tau_n\tau_{n+e_1+e_2}} \tag{41}$$

converts equations of motion (16) into the four-term bilinear Miwa equation:

$$v\tau_{n+e_1+e_2+e_3}\tau_n = u\tau_{n+e_1+e_2}\tau_{n+e_3} + w\tau_{n+e_2+e_3}\tau_{n+e_1} - uvw\tau_{n+e_1+e_3}\tau_{n+e_2}. \tag{42}$$

The 'D'-constraint (39) is just a real form on the equations of motion. For the Cauchy problem both these constraints mean the reduction of the number of degrees of freedom twice, $6N^2 \to 3N^2$. Also, the number of independent invariants of evolution is reduced nearly twice since

$$B, D: \quad T_{a,b} = T_{2N-a,2N-b}. \tag{43}$$

2.2. Quantum constraints: Fock space representation

Now we are back to the quantum world and the q-oscillator algebra (4). Here we consider the Fock space representation of q-oscillators over the Fock vacuum $|0\rangle$,

$$\boldsymbol{a}^-|0\rangle = 0, \qquad |n\rangle \sim \boldsymbol{a}^{+n}|0\rangle, \qquad \boldsymbol{k} = q^{\boldsymbol{N}+1/2}, \qquad \mathrm{N}|n\rangle = |n\rangle n. \tag{44}$$

Here N is the occupation number operator. If $0 < q < 1$ and $(\boldsymbol{a}^-)^\dagger = \boldsymbol{a}^+$, the R-matrix and evolution operators are unitary. Constraints B and D, equations (38) and (39), are conditions for states:

$$B: \quad u\boldsymbol{a}^+|\psi^B(u)\rangle = (1 - q^{-1/2}\boldsymbol{k})|\psi^B(u)\rangle \tag{45}$$

and

$$D: \quad (\boldsymbol{a}^- - u\boldsymbol{a}^+)|\psi^D(u)\rangle = 0. \tag{46}$$

Parameter u here is an extra useful $\mathbb{C}$-valued parameter making norms of $|\psi^B(u)\rangle$ and $|\psi^D(u)\rangle$ finite. Solutions to (45) and (46) are respectively

$$|\psi^B(u)\rangle = \sum_{n=0}^{\infty} \frac{(u\boldsymbol{a}^+)^n}{(q;q)_n}|0\rangle = \left(u\boldsymbol{a}_j^+;q\right)_\infty^{-1}|0\rangle, \tag{47}$$

and

$$|\psi^D(u)\rangle = \sum_{n=0}^{\infty} \frac{\left(u\boldsymbol{a}_j^{+2}\right)^n}{(q^4;q^4)_n}|0\rangle = \left(u\boldsymbol{a}_j^{+2};q^4\right)_\infty^{-1}|0\rangle. \tag{48}$$

In these formulae we use the Pochhammer symbol

$$(x;p)_n = (1-x)(1-px)\cdots(1-p^{n-1}x). \tag{49}$$

Norms of $|\psi^B\rangle$ and $|\psi^D\rangle$ are given by

$$\langle\psi^B(v)|\psi^B(u)\rangle = \frac{(-qvu;q)_\infty}{(vu;q)_\infty}, \qquad \langle\psi^D(v)|\psi^D(u)\rangle = \frac{(q^2vu;q^4)_\infty}{(vu;q^4)_\infty}. \tag{50}$$

Note, 'B'-relation (45) provides

$$\boldsymbol{a}^-|\psi^B(u)\rangle = u(1+q^{1/2}\boldsymbol{k})|\psi^B(u)\rangle. \tag{51}$$

Statement 1. There are two types of invariant subspaces of the R-matrix in the Fock space representations,

$$\mathsf{R}_{123}|\Omega\rangle = |\Omega\rangle. \tag{52}$$

They are

$$|\Omega\rangle = |\Omega^B\rangle = |\psi^B(u)\rangle_1 \otimes |\psi^B(uv)\rangle_2 \otimes |\psi^B(v)\rangle_3 \tag{53}$$

and

$$|\Omega\rangle = |\Omega^D\rangle = |\psi^D(u)\rangle_1 \otimes |\psi^D(uv)\rangle_2 \otimes |\psi^D(v)\rangle_3 \tag{54}$$

for arbitrary u and v.

Since $|\psi^D\rangle$ involves only even occupation numbers, $|\Omega^D\rangle$ is the eigenstate of $\mathsf{R}'_{123} = (-)^{N_2}\mathsf{R}_{123}$ which corresponds to the Euclidean rotation coefficients $X = \left(\begin{smallmatrix} \boldsymbol{k} & \boldsymbol{a}^+ \\ -\boldsymbol{a}^- & \boldsymbol{k}\end{smallmatrix}\right)$ with $\det X = 1$.

In the present paper [18] the reader can find a scenario how to use the vectors $|\psi^D(u)\rangle$ as the three-dimensional boundary states to reproduce the R-matrices, L-operators and representation structure of $\mathscr{U}_q\left(D_n^{(1)}\right)$. In a similar and even simpler way the vectors $|\psi^B(u)\rangle$ can be used as the boundary states reproducing $\mathscr{U}_q\left(B_n^{(1)}\right)$. However, the quantum groups exercises are not quite relevant to the study of the three-dimensional evolution operator.

Using formulae for map (6), we can instantly obtain

$$\mathsf{R}_{123}\boldsymbol{k}_2|\Omega^B\rangle = (\boldsymbol{k}_1 + \boldsymbol{k}_3 - \boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3 + (q^{1/2} - q^{-1/2})\boldsymbol{k}_1\boldsymbol{k}_3)|\Omega^B\rangle, \tag{55}$$

which is the quantum counterpart of (40). However, decomposition of $\mathsf{R}_{123}F(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)|\Omega^B\rangle$ for arbitrary function F is not well defined, and the modular representation is preferable.

2.3. Quantum constraints: modular representation

For the modular representation of the q-oscillator we use the Heisenberg pair σ, p,

$$[\sigma, p] = \frac{\mathrm{i}}{2\pi}. \tag{56}$$

The oscillator is given by

$$q = \mathrm{e}^{\mathrm{i}\pi b^2}, \qquad \boldsymbol{k} = -\mathrm{i}\,\mathrm{e}^{\pi b\sigma}, \qquad \boldsymbol{w} = \mathrm{e}^{2\pi bp}, \qquad \boldsymbol{a}^{\pm} = (1 - q^{\mp 1}\boldsymbol{k}^2)^{1/2}\boldsymbol{w}^{\pm 1}. \tag{57}$$

The dual one is then

$$\overline{q} = \mathrm{e}^{-\mathrm{i}\pi b^{-2}}, \qquad \overline{\boldsymbol{k}} = \mathrm{i}\,\mathrm{e}^{\pi b^{-1}\sigma}, \qquad \overline{\boldsymbol{w}} = \mathrm{e}^{2\pi b^{-1}p}, \qquad \overline{\boldsymbol{a}^{\pm}} = (1 - \overline{q}^{\mp 1}\overline{\boldsymbol{k}}^2)^{1/2}\overline{\boldsymbol{w}}^{\mp 1}. \tag{58}$$

Define a state $|\Phi\rangle$ by its wavefunction:

$$\langle\sigma|\Phi\rangle = \Phi(\sigma) = \exp\left(\frac{1}{8}\int_{\mathbb{R}_+} \frac{\mathrm{e}^{-2\mathrm{i}\sigma w}}{\sinh(bw)\cosh(b^{-1}w)}\frac{\mathrm{d}w}{w}\right). \tag{59}$$

The modular invariance $b \leftrightarrow b^{-1}$ is broken. The state $|\Phi\rangle$ satisfies

$$\boldsymbol{a}^{+}|\Phi\rangle = (1 - q^{-1/2}\boldsymbol{k}^2)|\Phi\rangle, \tag{60}$$

which is the B-type condition, and

$$(\overline{\boldsymbol{a}^{-}} - \overline{\boldsymbol{a}^{+}})|\Phi\rangle = 0, \tag{61}$$

which is the D-type condition. The asymptotic of Φ is

$$\Phi(\sigma)_{\sigma\to-\infty} \to 1, \qquad \Phi(\sigma)_{\sigma\to+\infty} \to \mathrm{e}^{-\pi b^{-1}\sigma/2}. \tag{62}$$

Let next

$$|\Phi_d\rangle = \boldsymbol{k}^d|\Phi\rangle, \tag{63}$$

where d is real (and integer). A test of asymptotic of function $\Phi(\sigma)$ shows that the state $|\Phi_d\rangle$ has a finite norm if

$$0 < d < \frac{1}{2b^2}. \tag{64}$$

Thus, in what follows we imply the quantum regime near quasi-classical point $b = 0$:

$$0 < b \ll 1, \tag{65}$$

so that d can be reasonably high.

Statement 2. For the modular representation of the q-oscillators operator R_{123} has the eigenstates (52) given by

$$|\Omega\rangle = |\Phi_d\rangle_1 \otimes |\Phi_{d+d'}\rangle_2 \otimes |\Phi_{d'}\rangle_3. \tag{66}$$

Moreover, due to the asymptotic of R, one can verify that if a state $F(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|\Omega\rangle$ has a finite norm, then the convolution $\mathsf{R}_{123}F(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|\Omega\rangle$ is convergent and therefore a map $F \to F'$ in the space of meromorphic functions with proper asymptotic (64)

$$\mathsf{R}_{123}F(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|\Omega\rangle = F'(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|\Omega\rangle \tag{67}$$

is well defined. This extends equation (55) to a subspace of whole Hilbert space–quantum BKP theory. In the classical limit $b \to 0$ ($q \to 1$) this map becomes the rational one,

$$F'(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) \underset{b\to 0}{\to} F(\boldsymbol{k}'_1, \boldsymbol{k}'_2, \boldsymbol{k}'_3), \tag{68}$$

where $\boldsymbol{k}'_j$ are given by (40).

3. Ground states of the evolution operator

3.1. Fock space representation

The straightforward extension of (53, 54) to the whole constant time surface is

$$|\Omega\rangle = \prod_{\boldsymbol{r}=n_1\boldsymbol{e}_x+n_3\boldsymbol{e}_y} |\psi(u_{n_3})\rangle_{1,\boldsymbol{r}} \otimes |\psi(u_{n_3}v_{n_1})\rangle_{2,\boldsymbol{r}} \otimes |\psi(v_{n_1})\rangle_{3,\boldsymbol{r}} \tag{69}$$

for B and D states in Fock space. The thus defined $|\Omega\rangle$ are the eigenstates of the evolution operator,

$$U|\Omega\rangle = |\Omega\rangle, \tag{70}$$

for arbitrary u_{n_3} and v_{n_1}. Series decomposition of $|\Omega\rangle$ gives an infinite set of eigenstates corresponding to fixed eigenvalues of

$$J_{n_3} = \prod_{n_1} \boldsymbol{k}_{1,n_1\boldsymbol{e}_x+n_3\boldsymbol{e}_y}\boldsymbol{k}_{2,n_1\boldsymbol{e}_x+n_3\boldsymbol{e}_y}, \qquad K_{n_1} = \prod_{n_3} \boldsymbol{k}_{2,n_1\boldsymbol{e}_x+n_3\boldsymbol{e}_y}\boldsymbol{k}_{3,n_1\boldsymbol{e}_x+n_3\boldsymbol{e}_y}. \tag{71}$$

The set of operators J and K belongs to the family of integrals of motion (36).

Remarkably, the thus constructed eigenstates of the evolution operator are not in general eigenstates of all integrals of motion. Thus, the more general form of evolution eigenstates is given by (u, v)-decomposition of

$$\prod_{a,b} T_{a,b}^{n_{a,b}}|\Omega\rangle, \tag{72}$$

where $T_{a,b}$ is defined by (36). Recall that $T_{a,b}|\Omega\rangle = T_{2N-a,2N-b}|\Omega\rangle$, in (72) we use a set of independent $T_{a,b}$ with non-diagonal action on $|\Omega\rangle$.

3.2. Modular representation

For the modular representation the basic eigenstate of the evolution operator is given by

$$|\Omega\rangle = \prod_{n_3} J_{n_3}^{d_{n_3}} \cdot \prod_{n_1} K_{n_1}^{d'_{n_1}} \cdot \prod_{\boldsymbol{r},j} |\Phi\rangle_{j,\boldsymbol{r}}, \tag{73}$$

where J and K are given by (71) and $|\Phi\rangle$ is given by (59). This state has the finite norm if

$$0 < d_{n_3}, \qquad d'_{n_1} < \frac{1}{2b^2}, \tag{74}$$

confer with (64). The extended set of eigenstates is given by (72). Note that the J, K pre-factor in (73) is an element of $\prod_{a,b} T_{a,b}^{n_{a,b}}$. For sufficiently small b the states (72) have finite norms.

However, contrary to the Fock space case, the states

$$\prod_{a,b} \overline{T}_{a,b}^{n_{a,b}}|\Omega\rangle, \tag{75}$$

where $\overline{T}_{a,b}$ are modular partners to $T_{a,b}$, do not have finite norms.

4. Conclusion

General evolution operators for three-dimensional field theories are given by (27),

$$\langle\boldsymbol{\sigma}|U|\boldsymbol{\sigma}'\rangle = \prod_{\boldsymbol{r}\in\mathbb{Z}_N^2} \langle\sigma_{1,\boldsymbol{r}}, \sigma_{2,\boldsymbol{r}}, \sigma_{3,\boldsymbol{r}}|\mathcal{R}|\sigma'_{1,\boldsymbol{r}+\boldsymbol{e}_x}, \sigma'_{2,\boldsymbol{r}}, \sigma'_{3,\boldsymbol{r}+\boldsymbol{e}_y}\rangle, \tag{76}$$

where the constant R-matrix is replaced by

$$\mathcal{R}_{123} = \varrho^{-1}\,\mathrm{e}^{\mathrm{i}(\pi-\phi_2)\boldsymbol{N}_2}\,\mathsf{R}_{123}\,\mathrm{e}^{-\mathrm{i}\phi_1\boldsymbol{N}_1-\mathrm{i}\phi_3\boldsymbol{N}_3} \tag{77}$$

for the Fock space representation, N_j here are the occupation numbers, and by

$$\mathcal{R}_{123} = \varrho^{-1}\, \mathrm{e}^{2\mathrm{i}\eta\phi_2\sigma_2}\, \mathsf{R}_{123}\, \mathrm{e}^{-2\mathrm{i}\eta\phi_1\sigma_1 - 2\mathrm{i}\eta\phi_3\sigma_3} \tag{78}$$

for the modular representation, η here is the crossing parameter, $\eta = \frac{1}{2}(b + b^{-1})$.

R-matrices and evolution operators are unitary for real spectral parameters ϕ_i. Quantum field theories have good quasi-classical limits for positive ϕ_i corresponding to sides of certain hyperbolic triangles [19]. Spectra of evolution operators essentially depend on values of spectral parameters. Presumably, the positiveness of the spectral parameters and a proper choice of the unitary normalization factor ϱ in (77) and (78) provide a good physical interpretation of the evolution spectra in terms of ground state and elementary excitations.

In this paper we consider the special case of trivial spectral parameters. The main result of the paper is the observation of essential degeneracy of the ground state $U = 1$ of the spectral parameters free case. The eigenstates constructed belong to a subspace of Hilbert space where Heisenberg evolution is a q-analogue of discrete BKP equations. These eigenstates however are not orthogonal and do not solve the problem of diagonalization of all integrals of motion.

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Appendix. Matrix elements and kernel

A.1. Matrix elements

Matrix elements of the R-matrix of statement 1 in the unitary Fock basis

$$F^{+}: \quad |n\rangle = \frac{\boldsymbol{a}^{+n}}{\sqrt{(q^2;q^2)_n}}|0\rangle, \qquad n \geqslant 0 \tag{A.1}$$

are given by

$$\langle n_1 n_2 n_3|\mathsf{R}|n_1' n_2' n_3'\rangle = \delta_{n_1+n_2,n_1'+n_2'}\delta_{n_2+n_3,n_2'+n_3'} \prod_{i=1}^{3} c_{n_i,n_i'}$$
$$\times\, q^{n_1 n_3 + n_2'} \frac{1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}z}{z^{n_2'+1}} \frac{(-q^{2+n_1'+n_3'}z;q^2)_\infty (-q^{-n_1-n_3}z;q^2)_\infty}{(-q^{+n_1-n_3}z;q^2)_\infty(-q^{-n_1+n_3}z;q^2)_\infty}, \tag{A.2}$$

where

$$c_{n,n'} = \sqrt{\frac{(q^2;q^2)_{n'}}{(q^2;q^2)_n}} \qquad \text{for} \quad n = 0, 1, 2, 3 \ldots. \tag{A.3}$$

Here Pochhammer's symbol and Euler's quantum dilogarithm are defined by (49). Coefficients $c_{n,n'}$ are just gauge factors. The *clockwise* integration loop in (A.2) circles all poles from dilogarithms but does not include $z = 0$. The Cauchy integral expression is equivalent to generating functions from [17].

Formula (A.2) serves in fact eight different R-matrices. The occupation numbers in (A.2) are in general integers,

$$n \in \mathbb{Z} = \mathbb{Z}_{<0} \oplus \mathbb{Z}_{\geqslant 0}. \tag{A.4}$$

This corresponds to the direct sum of Fock and anti-Fock representations,

$$F = F^{-} \oplus F^{+}. \tag{A.5}$$

Matrix (A.2) has the block-diagonal structure in

$$F_1^{\epsilon_1} \otimes F_2^{\epsilon_2} \otimes F_3^{\epsilon_3}, \qquad \epsilon_i = \pm. \tag{A.6}$$

The R-matrix is unitary in four blocks with $\epsilon_1\epsilon_2\epsilon_3 = +$. In anti-Fock components quantum constraints (45) and (46) should be slightly modified. In this paper we use the block $F_1^+ \otimes F_2^+ \otimes F_3^+$ where (A.2) is equivalent to the constant R-matrix from [1].

A.2. Kernel for modular representation

The kernel of R-matrix of statement 2 in representation (57,58) is given by [1]

$$\begin{aligned}\langle\sigma_1\sigma_2\sigma_3|\mathsf{R}|\sigma_1'\sigma_2'\sigma_3'\rangle &= \delta_{\sigma_1+\sigma_2,\sigma_1'+\sigma_2'}\delta_{\sigma_2+\sigma_3,\sigma_2'+\sigma_3'}\sqrt{\frac{\varphi(\sigma_1)\varphi(\sigma_2)\varphi(\sigma_3)}{\varphi(\sigma_1')\varphi(\sigma_2')\varphi(\sigma_3')}}\\ &\mathrm{e}^{-\mathrm{i}\pi(\sigma_1\sigma_3-\mathrm{i}\eta(\sigma_1+\sigma_3-\sigma_2'))}\int_{\mathbb{R}}\mathrm{d}u\,\mathrm{e}^{2\pi\mathrm{i}u(\sigma_2'-\mathrm{i}\eta)}\frac{\varphi\left(u+\frac{\sigma_1'+\sigma_3'+\mathrm{i}\eta}{2}\right)\varphi\left(u+\frac{-\sigma_1-\sigma_3+\mathrm{i}\eta}{2}\right)}{\varphi\left(u+\frac{\sigma_1-\sigma_3-\mathrm{i}\eta}{2}\right)\varphi\left(u+\frac{\sigma_3-\sigma_1-\mathrm{i}\eta}{2}\right)},\end{aligned} \tag{A.7}$$

where $\varphi(\sigma)$ is the Barns–Faddeev non-compact quantum dilogarithm [7]

$$\varphi(z) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+\mathrm{i}0}\frac{\mathrm{e}^{-2\mathrm{i}zw}}{\sinh(bw)\sinh(w/b)}\frac{\mathrm{d}w}{w}\right) \tag{A.8}$$

and $\eta = \frac{1}{2}(b+b^{-1})$ is the crossing parameter. In this paper we imply the regime of large crossing parameter, $0 < b \ll 1$. Both the quantum dilogarithm $\varphi(\sigma)$ and asymmetric function $\Phi(\sigma)$ (59) are analytical in the strip

$$-\eta < \mathrm{Im}(\sigma) < \eta. \tag{A.9}$$

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